Seeded Database Matching Under Noisy Column Repetitions

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New York University

Information Theory Workshop 2022

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- Are anonymized data truly private?
- NO!
  - Correlated public data $\rightarrow$ De-anonymization!

We Found Joe Biden's Secret Venmo. Here's Why That's A Privacy Nightmare For Everyone.

The peer-to-peer payments app leaves everyone from ordinary people to the most powerful person in the world exposed.


Ryan Mac
BuzzFeed News Reporter


Katie Notopoulos
BuzzFeed News Reporter


Ryan Brooks
BuzzFeed News Reporter


Logan McDonald
BuzzFeed Staff

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- Anonymized databases containing *micro-information* shared and published routinely.
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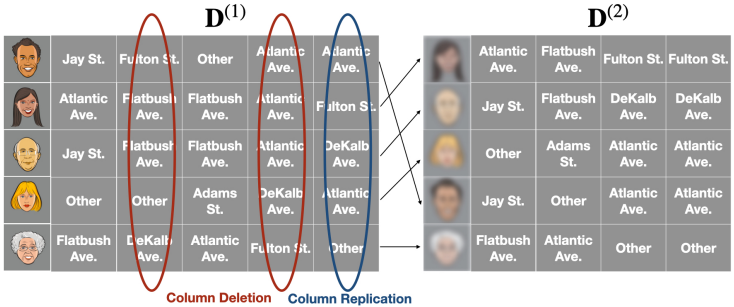
- This work: Time-indexed data, *e.g.*, financial and location data
- Synchronization errors in time-indexed data: column repetitions

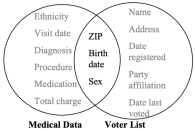


Practical Database Matching Attacks

- [Narayanan and Shmatikov, 2008]
  De-anonymization of Netflix Prize Database using IMDB data.

- [Sweeney, 2002]
  De-anonymization of medical databases using voter registration data.

- [Naini et al., 2012]
  User identification from geolocation data.



(a) Unlabeled histograms (Day 1)

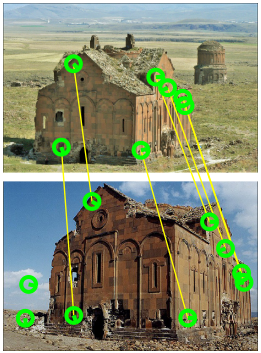
User	Location		
	Dorm.	Rest.	Lib.
?	75%	15%	10%
?	31%	30%	39%
?	15%	15%	70%
?	15%	65%	20%

(b) Labeled histograms (Day 2)

User	Location		
	Dorm.	Rest.	Lib.
John	33%	33%	34%
Jill	70%	20%	10%
Mary	15%	60%	25%
Mike	15%	20%	65%

Database Matching: Other Applications

- Computer vision [Galstyan et al., 2021]



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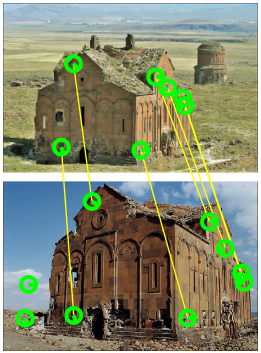

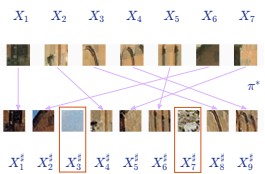

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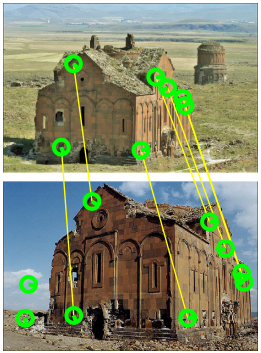

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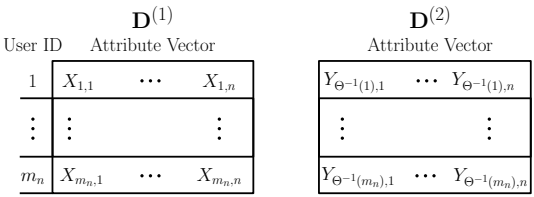
  - Single-cell data alignment [Chen et al., 2022]

Previous Works: Information-Theoretical Limits

[Shirani, Garg, and Erkip, ISIT 2019]



- Databases as $m_n \times n$ random matrices: equal no. of labeled attributes (columns)
  - Matching rows $\sim f_{X^{(1),n},X^{(2),n}}$

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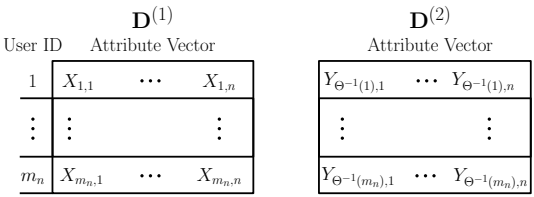

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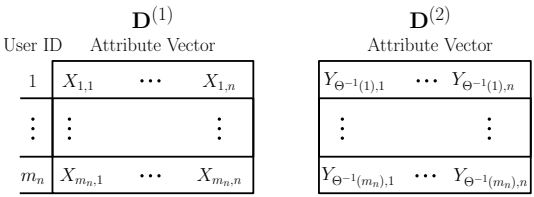

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  - Matching rows $\sim f_{X^{(1),n},X^{(2),n}}$
- Database growth rate: $R = \lim_{n\to\infty} \frac{1}{n} \log m_n$
- Successful matching: $P_e \to 0$ as $n \to \infty$
- Database matching $\Leftrightarrow$ Channel decoding

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- Objective: Given $(\mathbf{D}^{(1)}, \mathbf{D}^{(2)})$, find $\hat{\Theta}$ s.t.:

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- This allows us to
  - use information-theoretic tools,
  - work with arbitrary distributions.

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- Database Matching Under Column Deletions.
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- Extracting this side information from a batch of correctly-matched rows (seeds).

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[Bakirtas and Erkip, Asilomar 2022]

- Matching of Markov Databases Under Random Column Repetitions.
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    - Correlation among attributes
  - Noiseless setting.

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- Repetition detection is possible without seeds in the noiseless setting.
- Complete characterization of the matching capacity in the noiseless setting.

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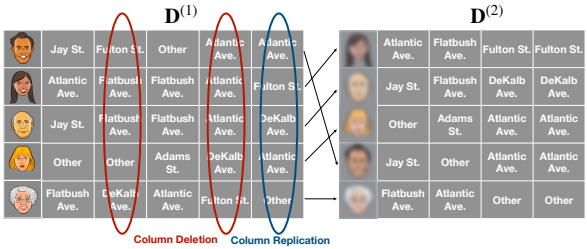
5. The retaining entries are noisy.



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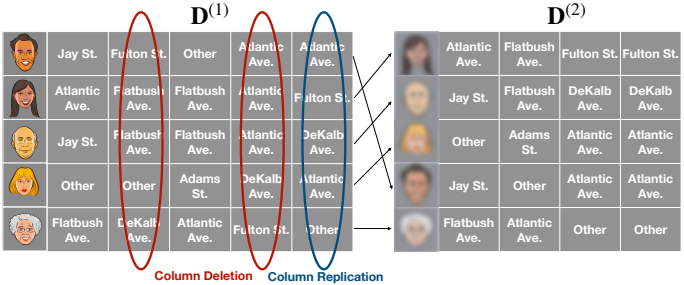

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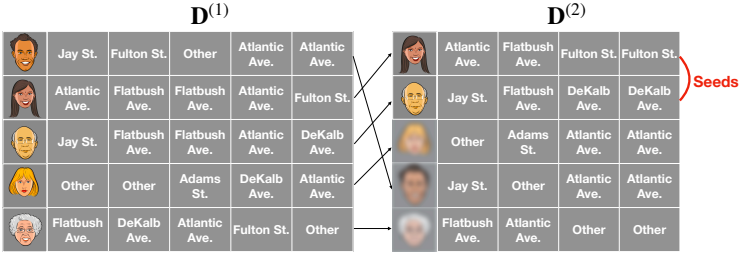

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- Column repetition pattern: random vector $S^n = \{S_1, S_2, ..., S_n\}$ with $S_j \overset{i.i.d.}{\sim} p_S$.
  - $supp(p_S) = \{0, \ldots, s_{\max}\}$

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   - X and Y are not independent: $p_{Y|X} \neq p_Y$

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- Goal: Given $p_X, p_{Y|X}, p_S, d$, characterize matching capacity $C(d)$.

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3. Can we extract the repetition pattern from seeds?
4. If yes, how many seeds are sufficient?

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- We will use the *Hamming distances between the consecutive columns of* $\mathbf{D}^{(2)}$ as the permutation-invariant feature.

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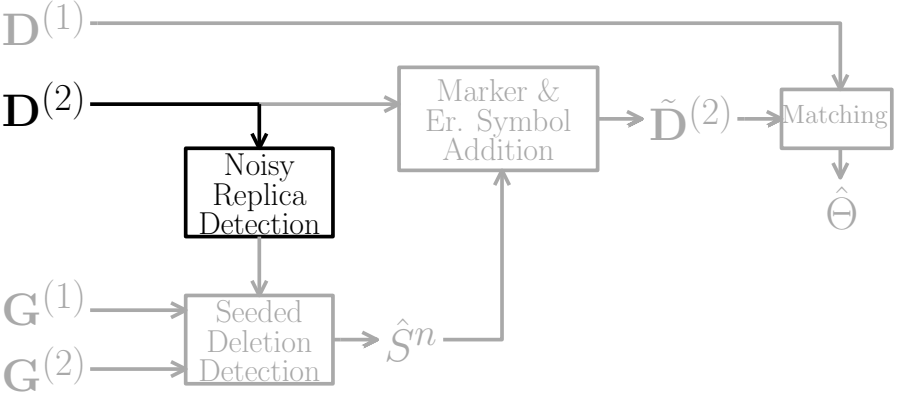

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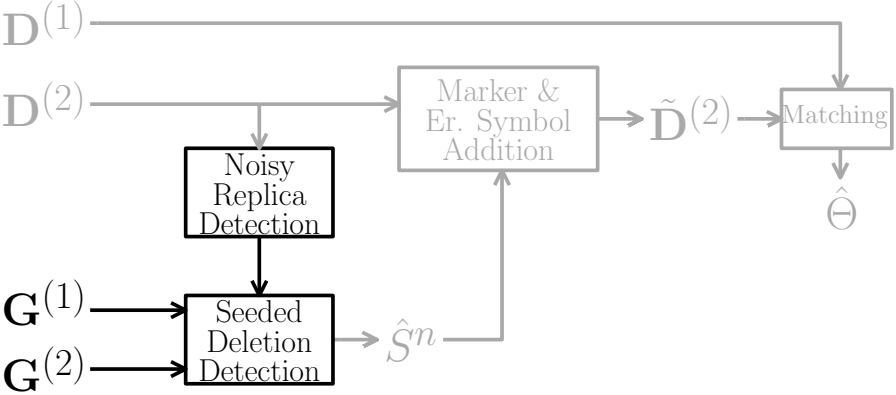

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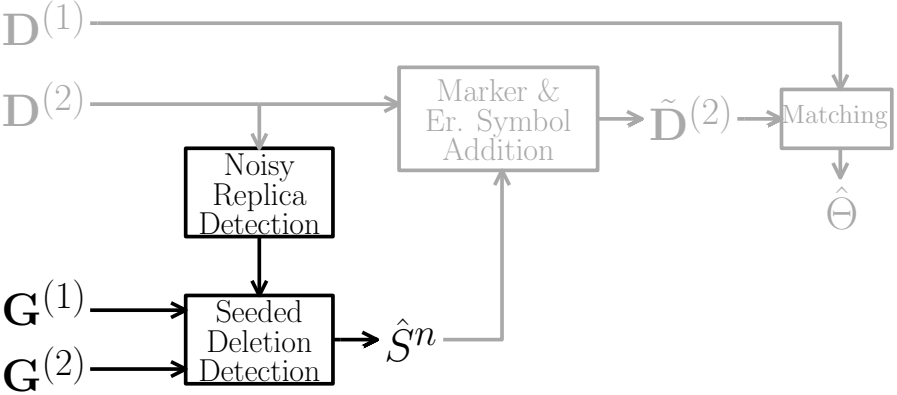

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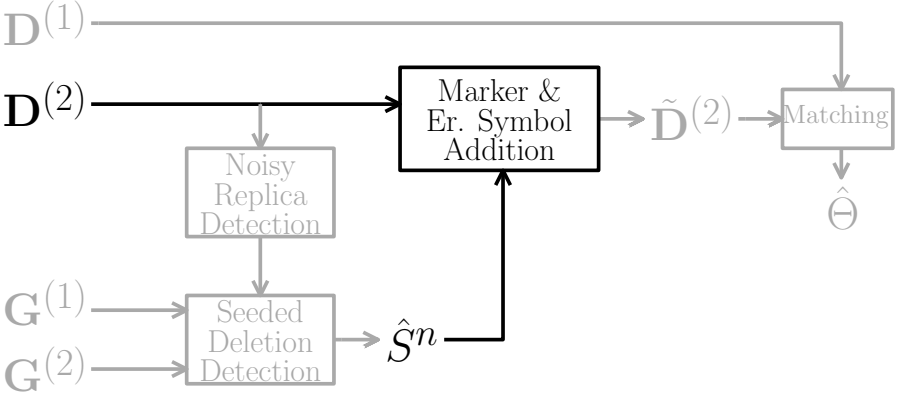

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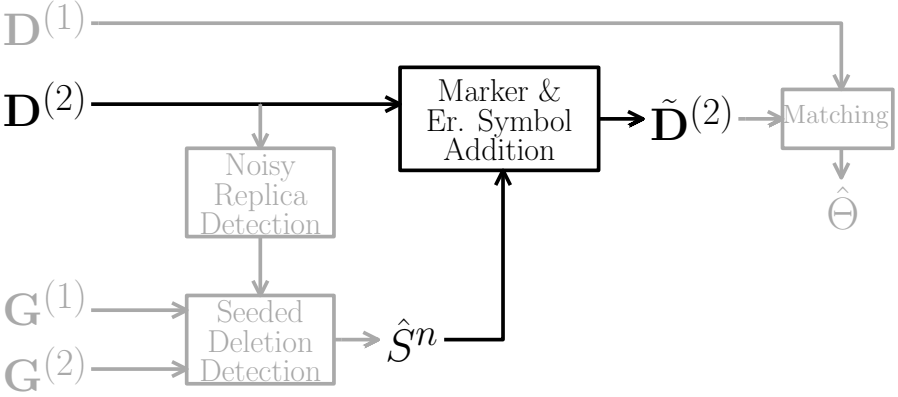

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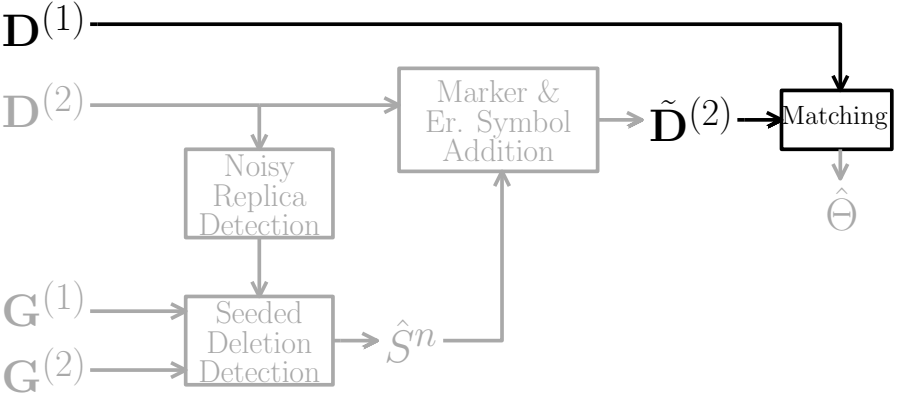

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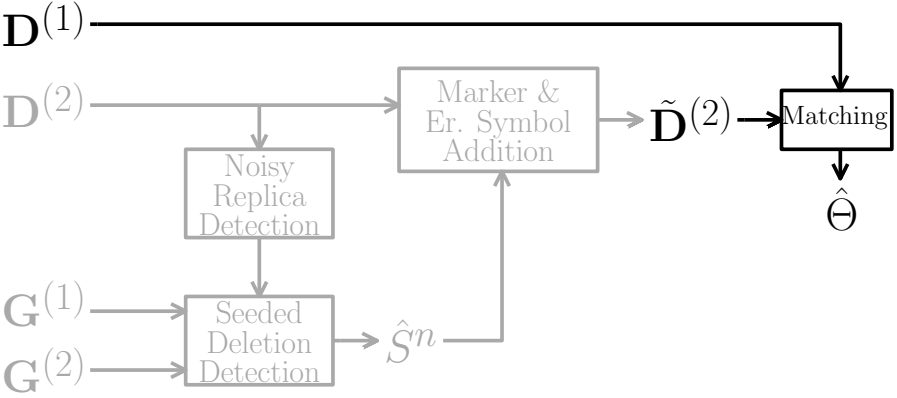

Noisy Replica Detection

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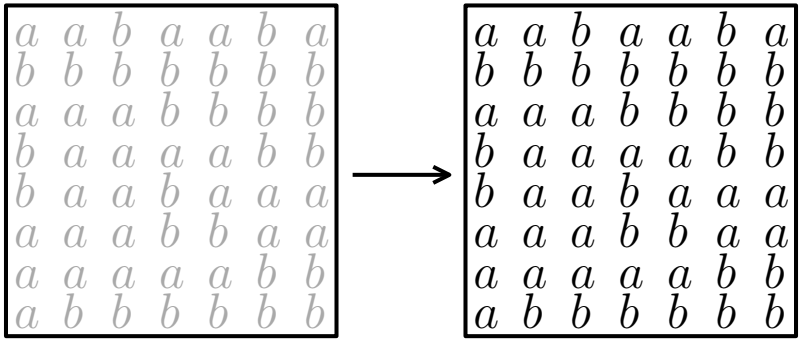
3. Declare $C_j^{m_n}$ and $C_{j+1}^{m_n}$ to be
   - noisy replicas, if $d_H(C_j^{m_n}, C_{j+1}^{m_n}) < m_n\tau$.
   - independent, if $d_H(C_j^{m_n}, C_{j+1}^{m_n}) \geq m_n\tau$.

Noisy Replica Detection: Example

$\mathbf{D}^{(2)}$ $\quad m_n\tau = 2 \quad$ $\mathbf{D}^{(2)}$



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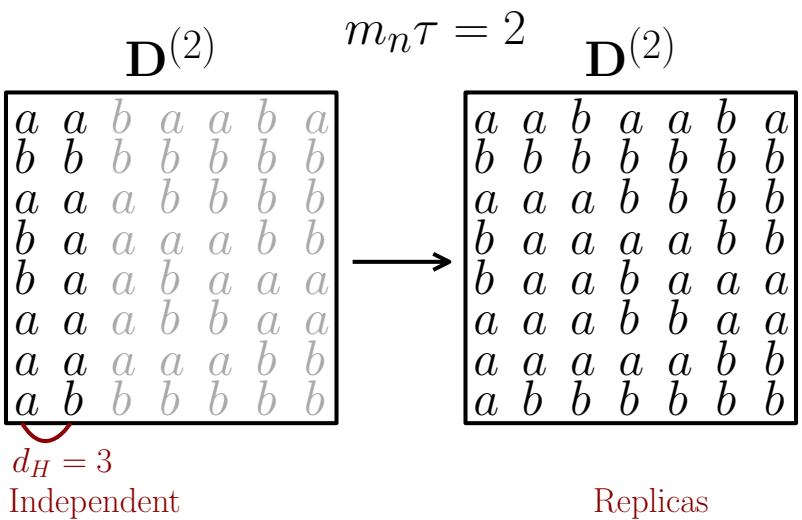

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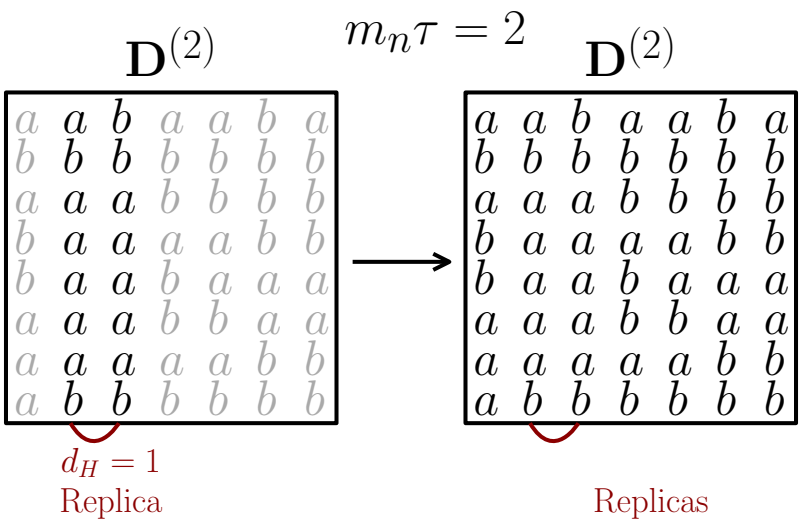

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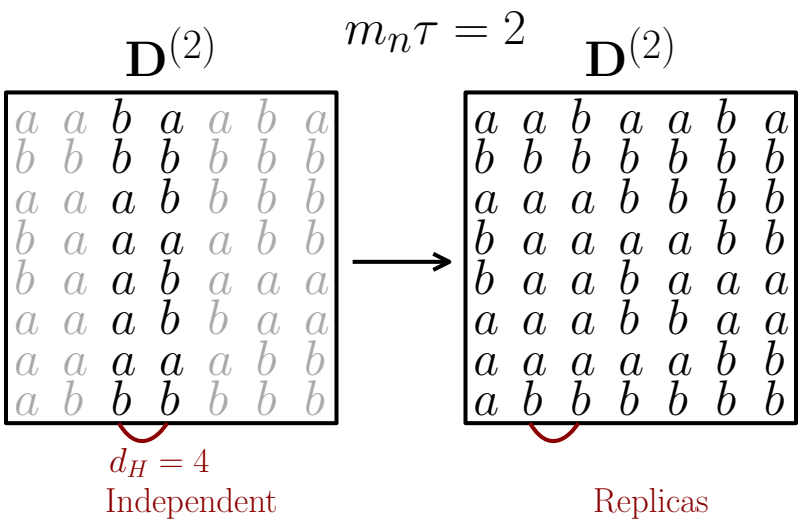

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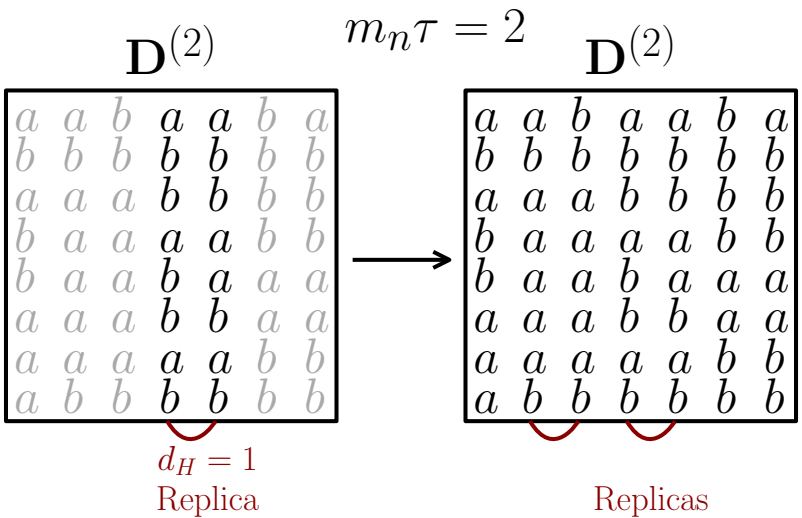

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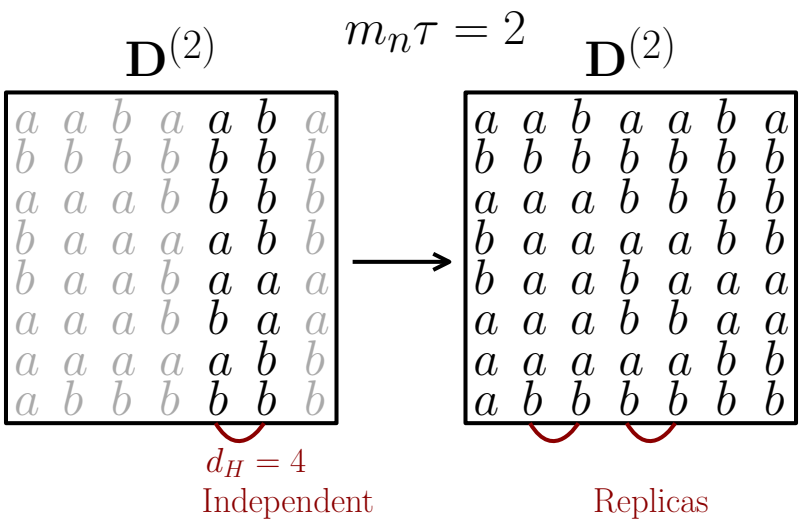

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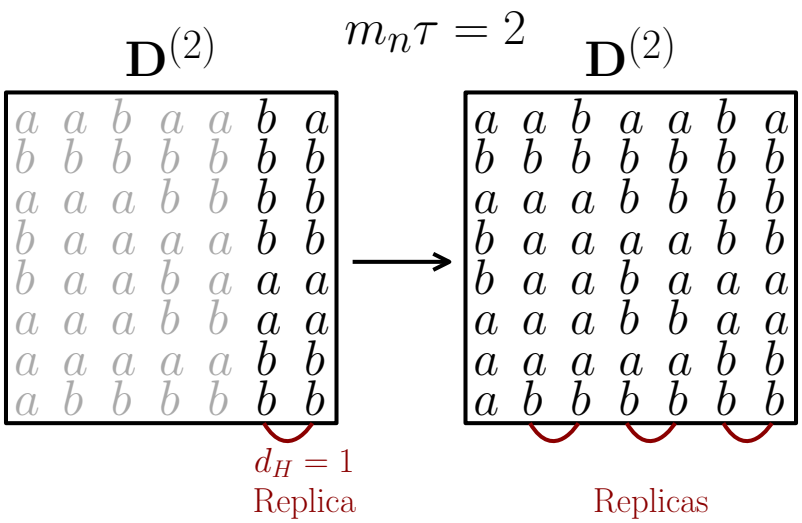

Noisy Replica Detection

Lemma

Let E_j denote the event that the aforementioned Hamming distance based algorithm fails to infer the correct relationship between the columns $C_j^{m_n}$ and $C_{j+1}^{m_n}$ of $\mathbf{D}^{(2)}$, $j = 1, \ldots, K-1$. Then

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- m_n being exponential in n is enough.

Lemma: Sketch of Proof

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  - $\text{Binom}(m_n, p_1)$ if $C_j^{m_n}$ and $C_{j+1}^{m_n}$ are noisy replicas.
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- Apply Chernoff bound to the summands
- $\Theta(n)$ summands, each decaying exponentially with m_n.

Seeded Deletion Detection Algorithm

1. Perform noisy replica detection on $\mathbf{D}^{(2)}$.
2. Discard all-but-one of the replicas from $\mathbf{G}^{(2)}$ to obtain $\tilde{\mathbf{G}}^{(2)}$.
3. If necessary, apply a mapping Φ to the entries of $\tilde{\mathbf{G}}^{(2)}$ to obtain $\tilde{\mathbf{G}}^{(2)}_{\Phi}$
   - Φ satisfies

$$\Pr(\Phi(Y_1) \neq X_2) > \Pr(\Phi(Y_1) \neq X_1)$$

4. Perform an exhaustive search over all potential deletion patterns on $\mathbf{G}^{(1)}$.
5. For each deletion pattern I, compute the total Hamming distance $d_H(\tilde{\mathbf{G}}^{(1)}_I, \tilde{\mathbf{G}}^{(2)}_{\Phi})$ between $\tilde{\mathbf{G}}^{(1)}_I$ and $\tilde{\mathbf{G}}^{(2)}_{\Phi}$.
6. Output the deletion pattern $\hat{I}_{\text{del}}(\Phi)$, minimizing total Hamming distance between $\tilde{\mathbf{G}}^{(1)}_I$ and $\tilde{\mathbf{G}}^{(2)}_{\Phi}$

$$\hat{I}_{\text{del}}(\Phi) = \underset{I \subseteq [n]}{\arg\min}\, d_H(\tilde{\mathbf{G}}^{(1)}_I, \tilde{\mathbf{G}}^{(2)}_{\Phi})$$

Seeded Deletion Detection

Lemma

Let I_{del} be the underlying deletion pattern. Then there exists a bijective mapping Φ depending on $p_{X,Y}$ and for seed size $\Lambda_n \geq cnH_b(\delta)$,

$$\Pr\left(\hat{I}_{\text{del}}(\Phi) = I_{\text{del}}\right) \to 1 \text{ as } n \to \infty$$

where H_b denotes the binary entropy function, δ is the column deletion probability and c depends on $p_{X,Y}$.

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- A seed size linear with the column size n is sufficient!
- *i.e.*, a seed size logarithmic with the row size m_n is sufficient!

Lemma: Sketch of Proof

Detection:

- Union bound over all deletion patterns

$$\Pr\left(\hat{I}_{\text{del}}(\Phi) \neq I_{\text{del}}\right) \leq \sum_{I \subseteq [n], |I| = \hat{K}} \Pr(d_H(\tilde{\mathbf{G}}_I^{(1)}, \tilde{\mathbf{G}}_\Phi^{(2)}) \leq d_H(\tilde{\mathbf{G}}_{I_{\text{del}}}^{(1)}, \tilde{\mathbf{G}}_\Phi^{(2)}))$$

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- Observe

$$d_H(\tilde{\mathbf{G}}_I^{(1)}, \tilde{\mathbf{G}}_\Phi^{(2)}) - d_H(\tilde{\mathbf{G}}_{I_{\text{del}}}^{(1)}, \tilde{\mathbf{G}}_\Phi^{(2)}) = M - N$$

$$M \sim \text{Binom}(\Lambda_n(\hat{K} - f(I, I_{\text{del}})), q_0(\Phi))$$

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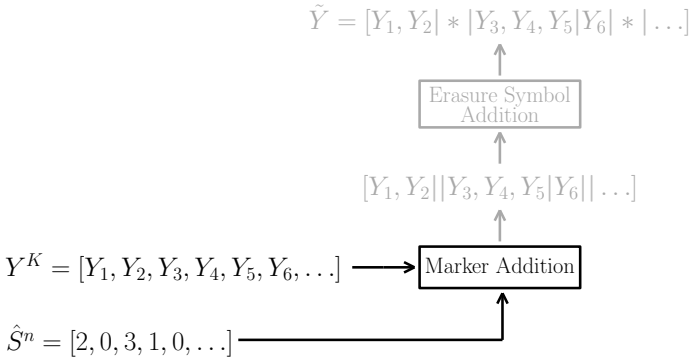
Marker & Erasure Symbol Addition

Y^K: a row of $\mathbf{D}^{(2)}$.



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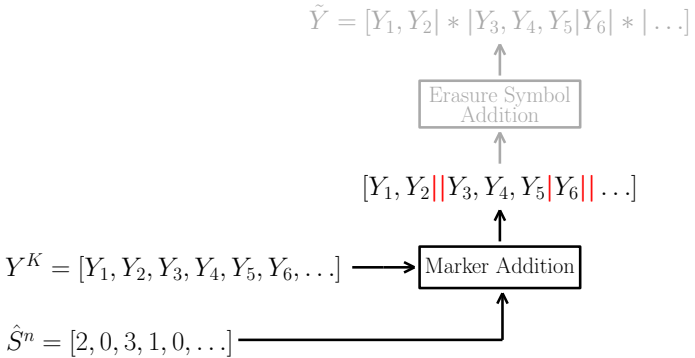

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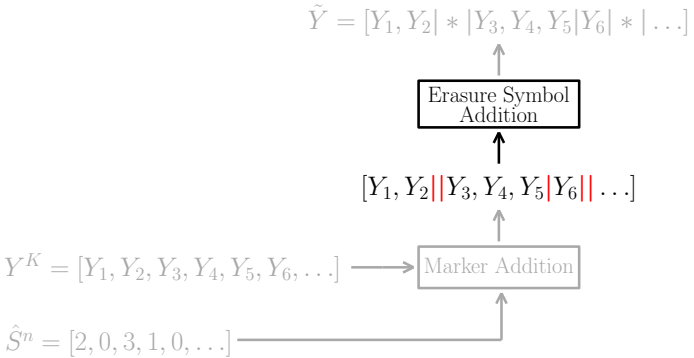

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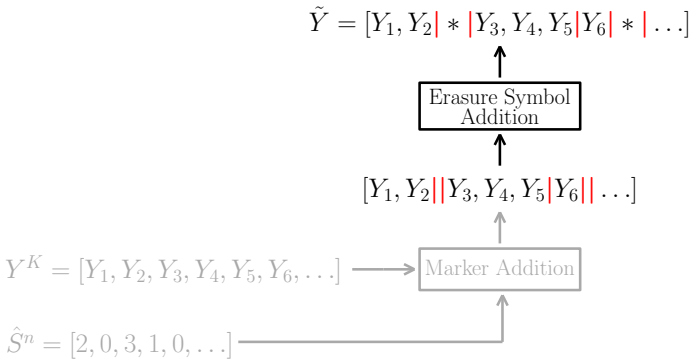

$\tilde{Y}$: the corresponding row of $\tilde{\mathbf{D}}^{(2)}$.

Main Result

Theorem: Main Result

Given a database distribution p_X, a column repetition distribution p_S and a noise distribution $p_{Y|X}$, for any seed order $d \geq 1$, the matching capacity is

$$C(d) = I(X; Y^S, S)$$

where $S \sim p_S$ and $Y^S = Y_1, \ldots, Y_S$ such that

$$\Pr(Y^S = y_1, \ldots, y_S | X = x) = \prod_{i=1}^{S} p_{Y|X}(y_i|x)$$

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  - Replication acts as a repetition code
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- We have a complete characterization of the matching capacity.

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  - Database matching when the repetition pattern is not constant across rows.

Thank you! Q&A?

Seeded Database Matching Under Noisy Column Repetitions

Serhat Bakirtas, Elza Erkip

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- This is not true in general!

Seeded Deletion Detection: Example

$n = 6$, $\Lambda_n = 8$. $\mathfrak{X} = \{a, b\}$, $p_X(a) = p_X(b) = 0.5$, $p_{Y|X} \sim \text{BSC}(q)$, $q = 0.75$.

$$I_{del} = [1,\ 3]$$

$\mathbf{G}^{(1)}$

b	b	a	b	b	a
b	b	a	b	a	a
a	b	a	b	b	b
a	a	a	b	a	b
a	a	a	a	a	a
a	a	a	b	b	a
b	b	b	a	b	a
b	b	a	b	a	a

$\tilde{\mathbf{G}}^{(2)}$

b	b	b	a
b	b	a	a
b	b	b	b
a	b	a	b
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b	b	a	b	a	a

$\tilde{\mathbf{G}}^{(2)}$

b	b	b	a
b	b	a	a
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b	b	b	a	b	a
b	b	a	b	a	a

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a	b	a	b
a	a	b	b
a	a	a	a
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b	b	a	b	a	a
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a	a	a	b	b	a
b	b	b	a	b	a
b	b	a	b	a	a

$\tilde{\mathbf{G}}^{(2)}$

a	b	a	b
a	a	b	b
a	a	a	a
b	a	b	a
b	a	b	b
a	a	a	b
a	b	a	b
a	a	b	b

$$\hat{I}_{del} = [2,\ 6]$$

Seeded Deletion Detection: Example

$n = 6$, $\Lambda_n = 8$. $\mathfrak{X} = \{a, b\}$, $p_X(a) = p_X(b) = 0.5$, $p_{Y|X} \sim \text{BSC}(q)$, $q = 0.75$.

$$I_{del} = [1,\ 3]$$

$\mathbf{G}^{(1)}$

b	b	a	b	b	a
b	b	a	b	a	a
a	b	a	b	b	b
a	a	a	b	a	b
a	a	a	a	a	a
a	a	a	b	b	a
b	b	b	a	b	a
b	b	a	b	a	a

$\tilde{\mathbf{G}}^{(2)}$

a	b	a	b
a	a	b	b
a	a	a	a
b	a	b	a
b	a	b	b
a	a	a	b
a	b	a	b
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$$\hat{I}_{del} = [2,\ 6]$$

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- $\hat{I}_{\text{del}} \neq I_{\text{del}}$ in the above example.

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- After applying Φ, we can use the aforementioned algorithm.

Seeded Deletion Detection: Example

$$I_{del} = [1,\ 3]$$

$\mathbf{G}^{(1)}$

b	b	a	b	b	a
b	b	a	b	a	a
a	b	a	b	b	b
a	a	a	b	a	b
a	a	a	a	a	a
a	a	a	b	b	a
b	b	b	a	b	a
b	b	a	b	a	a

$\tilde{\mathbf{G}}^{(2)}$

a	b	a	b
a	a	b	b
a	a	a	a
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b	a	b	b
a	a	a	b
a	b	a	b
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$$\hat{I}_{del}(\Phi) = [1,\ 3]$$

Seeded Deletion Detection: Example

$$I_{del} = [1,\ 3]$$

$\mathbf{G}^{(1)}$

b	b	a	b	b	a
b	b	a	b	a	a
a	b	a	b	b	b
a	a	a	b	a	b
a	a	a	a	a	a
a	a	a	b	b	a
b	b	b	a	b	a
b	b	a	b	a	a

$\tilde{\mathbf{G}}_{\Phi}^{(2)}$

b	a	b	a
b	b	a	a
b	b	b	b
a	b	a	b
a	b	a	a
b	b	b	a
b	a	b	a
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Seeded Deletion Detection: Example

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$\mathbf{G}^{(1)}$

b	b	a	b	b	a
b	b	a	b	a	a
a	b	a	b	b	b
a	a	a	b	a	b
a	a	a	a	a	a
a	a	a	b	b	a
b	b	b	a	b	a
b	b	a	b	a	a

$\tilde{\mathbf{G}}_{\Phi}^{(2)}$

b	a	b	a
b	b	a	a
b	b	b	b
a	b	a	b
a	b	a	a
b	b	b	a
b	a	b	a
b	b	a	a

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Seeded Deletion Detection: Example

$$I_{del} = [1,\ 3]$$

$\mathbf{G}^{(1)}$

b	b	a	b	b	a
b	b	a	b	a	a
a	b	a	b	b	b
a	a	a	b	a	b
a	a	a	a	a	a
a	a	a	b	b	a
b	b	b	a	b	a
b	b	a	b	a	a

$\tilde{\mathbf{G}}_{\Phi}^{(2)}$

b	a	b	a
b	b	a	a
b	b	b	b
a	b	a	b
a	b	a	a
b	b	b	a
b	a	b	a
b	b	a	a

$$\hat{I}_{del}(\Phi) = [1,\ 3]$$

Lemma: Sketch of Proof

Existence of Φ with desired property:

- Let

$$q_0(\Phi) \triangleq \Pr(\Phi(Y_1) \neq X_2)$$
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- Consider several one-cycle permutations over $\mathfrak{X}$ to show that

$$q_0(\Phi) - q_1(\Phi) = 0 \; \forall \Phi \Leftrightarrow p_{Y|X}(y|x) = p_Y(y) \; \forall (x, y) \in \mathfrak{X}^2$$

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- Thus, as long as $p_{Y|X} \neq p_Y$,

$$\exists \Phi \; q_0(\Phi) > q_1(\Phi)$$